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MIXING ANGLES OF QUARKS AND LEPTONS AS AN OUTCOME OF $SU(2)$ HORIZONTAL SYMMETRIES

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Abstract: We show that all mixing angles are determined, within experimental uncertainty, by a product of $SU(2)$ horizontal symmetries intimately linked to the algebra of weak neutral currents. This concerns: on one hand, the three quark mixing angles; on the other hand, a neutrino-like pattern in which θ_{23} is maximal and $\tan(2\theta_{12}) = 2$. The latter turns out to exactly satisfy the “quark-lepton complementarity condition” $\theta_c + \theta_{12} = \pi/4$. Moreover, among all solutions, two values for the third neutrino mixing angle arise, which satisfy the bound $\sin^2(\theta_{13}) \leq 0.1$: $\theta_{13} = \pm 5.7 \cdot 10^{-3}$ and $\theta_{13} = \pm 0.2717$.

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1 Introduction

Following the study of neutral kaons done in [1], we have shown in [2] and [3] that:

* in Quantum Field Theory (QFT), mixing matrices linking flavour to mass eigenstates for non-degenerate coupled systems should never be parametrized as unitary. Indeed, assuming that the effective renormalized quadratic Lagrangian is hermitian at any q^2 and that flavour eigenstates form an orthonormal basis, different mass eigenstates, which correspond to different values of q^2 (poles of the renormalized propagator) belong to different orthonormal bases ¹;

* when it is so, the properties of universality for diagonal neutral currents and absence of flavor changing neutral currents (FCNC) which are systematically implemented, for the Standard Model (SM), in the space of flavour eigenstates, do not automatically translate anymore into equivalent properties in the space of mass eigenstates. In the case of two generations of fermions, imposing them for mass eigenstates yields two types of solutions for the mixing angles ² of each doublet with identical electric charge: Cabibbo-like solutions ³ which reduce to a single unconstrained mixing angle, and a set of discrete solutions, unnoticed in the customary approach, including in particular the so-called maximal mixing $\pi/4 \pm k\pi/2$;

* for any of these solutions one recovers a unitary mixing matrix; but, as said above, very small deviations are expected due to mass splittings, which manifest themselves as a tiny departure from the exact two previous conditions. In particular, in the *neighborhood* of a Cabibbo-like solution, these deviations become of equal strength for a value of the mixing angle extremely close to the measured Cabibbo angle

$$\tan(2\theta_c) = 1/2. \quad (1)$$

This success was an encouragement to go further in this direction. We present below the outcome of our investigation of neutral current patterns in the case of three generations of fermions. In particular, we show that the requested scheme of unitarity violation of the mixing matrices trivially interprets in terms of a product of $SU(2)$ horizontal symmetries, implemented at the level of neutral weak currents. Hence, the values of all mixing angles, in the quark as well as in the leptonic sector, are controlled by this symmetry.

The intricate system of trigonometric equations has been solved by successive approximations, starting from configurations in which θ_{13} is vanishing. We will see that this approximation, obviously inspired by the patterns of mixing angles determined from experimental measurements, turns out to be a very good one. Indeed, we precisely show, without exhibiting all the solutions of our equations, that the presently observed patterns of quarks as well as of neutrinos, do fulfill our criterion. While the three angles of the Cabibbo-Kobayashi-Maskawa (CKM) solution are “Cabibbo-like”, the neutrino-like solution

$$\begin{aligned} \tan(2\theta_{12}) &= 2 \Leftrightarrow \theta_{12} \approx 31.7^\circ, \\ \theta_{23} &= \pi/4, \\ \theta_{13} &= \pm 5.7 \cdot 10^{-3} \text{ or } \theta_{13} = \pm 0.2717 \end{aligned} \quad (2)$$

is of a mixed type, where θ_{23} is maximal while θ_{12} and θ_{13} are Cabibbo-like.

Two significant features in these results must be stressed. First, the values for the third neutrino mixing angle θ_{13} given in (2) are predictions which take into account present (loose) experimental constraints. Only two possibilities survive: an extremely small value $\theta_{13} \sim V_{ub} \sim \text{a few } 10^{-3}$, and a rather “large” one, at the opposite side of the allowed range. Secondly, our procedure yields in an exact, though quite simple way, the well-known “quark-lepton complementarity relation” [6] for 1-2 mixing:

$$\theta_{12} + \theta_c = \pi/4, \quad (3)$$

where θ_{12} is the leptonic angle, and θ_c the Cabibbo angle for quarks.

¹Since at any *given* q^2 , the set of eigenstates of the renormalized quadratic Lagrangian form an orthonormal basis, the mixing matrix with all its elements evaluated at this q^2 is unitary and the unitarity of the theory is never jeopardized.

²For two generations, one is led to introduce two mixing angles to parametrize each 2×2 non-unitary mixing matrix.

³Cabibbo-like angles can only be fixed by imposing conditions on the *violation pattern* of the unitarity of the mixing matrix in its vicinity.

2 Neutral currents of mass eigenstates and $SU(2)$ symmetries

2.1 The different basis of fermions

Three bases will appear throughout the paper:

* flavour eigenstates, that we note (u_f, c_f, t_f) and (d_f, s_f, b_f) for quarks, (e_f, μ_f, τ_f) and $(\nu_{ef}, \nu_{\mu f}, \nu_{\tau f})$ for leptons;

* mass eigenstates that we note (u_m, c_m, t_m) and (d_m, s_m, b_m) for quarks, (e_m, μ_m, τ_m) and $(\nu_{em}, \nu_{\mu m}, \nu_{\tau m})$ for leptons; they include in particular the charged leptons detected experimentally, since their identification proceeds through the measurement of their *charge/mass* ratio in a magnetic field;

* the neutrinos that couple to mass eigenstates of charged leptons in charged weak currents. These are the usual "electronic", "muonic" and " τ " neutrinos ν_e, ν_μ, ν_τ considered in SM textbooks [5]: they are indeed identified by the outgoing charged leptons that they produce through charged weak currents, and the latter are precisely mass eigenstates (see above). These states read (see Appendix D)

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = K_\ell^\dagger \begin{pmatrix} \nu_{ef} \\ \nu_{\mu f} \\ \nu_{\tau f} \end{pmatrix} = (K_\ell^\dagger K_\nu) \begin{pmatrix} \nu_{em} \\ \nu_{\mu m} \\ \nu_{\tau m} \end{pmatrix}, \quad (4)$$

where K_ℓ and K_ν are the mixing matrices respectively of charged leptons and of neutrinos (*i.e.* the matrices that connect their flavour to their mass eigenstates). Note that these neutrinos coincide with flavour eigenstates when the mixing matrix of *charged leptons* is taken equal to unity $K_\ell = 1$, *i.e.* when the mass and flavour eigenstates of charged leptons are aligned, which is often assumed in the literature.

2.2 Neutral currents and $SU(2)$ symmetry

The principle of the method is best explained in the case of two generations. This in particular makes the link with our previous work [3].

Let us consider for example the (d, s) channel ⁴. The corresponding neutral currents in the basis of mass eigenstates are controlled by the product $C^\dagger C$ of the mixing matrix C between d and s with its hermitian conjugate (see [2]). Requesting the absence of non-diagonal currents and universality for diagonal currents (that we call hereafter the "unitarization" conditions) selects two classes of mixing angles [2]: a first class that we call "Cabibbo-like" which is unconstrained at this level, and a second class made of a series of discrete values, maximal or vanishing. As soon as C departs from unitarity, the Lagrangian for neutral currents reads, in the basis of mass eigenstates

$$\mathcal{L} \propto W_\mu^3 [\alpha \bar{d}_m \gamma_L^\mu d_m + \beta \bar{s}_m \gamma_L^\mu s_m + \delta \bar{d}_m \gamma_L^\mu s_m + \zeta \bar{s}_m \gamma_L^\mu d_m]. \quad (5)$$

The condition that was shown in [3] to fix the value of the "Cabibbo" angle, *i.e.* that the universality of $\bar{d}_m \gamma_L^\mu d_m$ and $\bar{s}_m \gamma_L^\mu s_m$ currents is violated with the same strength as the absence of $\bar{d}_m \gamma_L^\mu s_m$ and $\bar{s}_m \gamma_L^\mu d_m$ currents, now simply reads $\delta = \alpha - \beta = \zeta$, which states that \mathcal{L} in (5) is invariant by the $SU(2)$ symmetry which rotates d and s . Eq. (5) indeed trivially rewrites, then,

$$\begin{aligned} \mathcal{L} \propto W_\mu^3 & \left[(\alpha + \beta) \frac{\bar{d}_m \gamma_L^\mu d_m + \bar{s}_m \gamma_L^\mu s_m}{2} \right. \\ & \left. + (\alpha - \beta) \left(\frac{\bar{d}_m \gamma_L^\mu d_m - \bar{s}_m \gamma_L^\mu s_m}{2} + \bar{d}_m \gamma_L^\mu s_m + \bar{s}_m \gamma_L^\mu d_m \right) \right], \quad (6) \end{aligned}$$

⁴"Channel (i, j) " corresponds to two fermions i and j with identical electric charge; for example, "channel $(2, 3)$ " corresponds to (d, b) , (c, t) , (μ^-, τ^-) or (ν_μ, ν_τ) .

in which all components of the triplet of $SU(2)$ currents $[\frac{1}{2}(\bar{d}_m\gamma_L^\mu d_m - \bar{s}_m\gamma_L^\mu s_m), \bar{d}_m\gamma_L^\mu s_m, \bar{s}_m\gamma_L^\mu d_m]$, the corresponding (vector) charges of which make up the relevant $SU(2)$ algebra, have the same coefficient $(\alpha - \beta)$. The work [3] states accordingly that the “Cabibbo angle” is controlled by this $SU(2)$ symmetry.

The generalization to three generations is now straightforward. Neutral currents are controlled by the product $K^\dagger K$ of the 3×3 mixing matrix K with its hermitian conjugate; for example, the (left-handed) neutral currents for quarks with electric charge $(-1/3)$ read

$$\overline{\begin{pmatrix} d_f \\ s_f \\ b_f \end{pmatrix}} \gamma_L^\mu \begin{pmatrix} d_f \\ s_f \\ b_f \end{pmatrix} = \overline{\begin{pmatrix} d_m \\ s_m \\ b_m \end{pmatrix}} \gamma_L^\mu K_d^\dagger K_d \begin{pmatrix} d_m \\ s_m \\ b_m \end{pmatrix}. \quad (7)$$

Requesting $SU(2)$ symmetry in each (i, j) channel is trivially equivalent to the condition that, in this channel, universality for the diagonal currents is violated with the same strength as the absence of non-diagonal currents. We will show that all presently known mixing angles, in the quark as well as in the leptonic sectors, satisfy this criterion.

2.3 Mixing matrices. Notations

We write each mixing matrix K as a product of three matrices, which reduce, in the unitarity limit, to the basic rotations by θ_{12} , θ_{23} and θ_{13} (we are not concerned with CP violation)

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -\tilde{s}_{23} & \tilde{c}_{23} \end{pmatrix} \times \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -\tilde{s}_{13} & 0 & \tilde{c}_{13} \end{pmatrix} \times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\tilde{s}_{12} & \tilde{c}_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (8)$$

We parametrize each basic matrix, which is *a priori* non-unitary, with two angles, respectively $(\theta_{12}, \tilde{\theta}_{12})$, $(\theta_{23}, \tilde{\theta}_{23})$ and $(\theta_{13}, \tilde{\theta}_{13})$. We deal accordingly with six mixing angles, instead of three in the unitary case (where $\theta_{ij} = \tilde{\theta}_{ij}$). We will use throughout the paper the notations $s_{ij} = \sin(\theta_{ij})$, $\tilde{s}_{ij} = \sin(\tilde{\theta}_{ij})$, and likewise, for the cosines, $c_{ij} = \cos(\theta_{ij})$, $\tilde{c}_{ij} = \cos(\tilde{\theta}_{ij})$.

To lighten the text, the elements of $K^\dagger K$ will be abbreviated by $[ij]$, $i, j = 1 \dots 3$ instead of $(K^\dagger K)_{[ij]}$, and the corresponding neutral current will be noted $\{ij\}$. So, in the quark case, $\{12\}$ stands for $\bar{u}_m\gamma_L^\mu c_m$ or $\bar{d}_m\gamma_L^\mu s_m$, and, in the neutrino case, for $\bar{\nu}_{em}\gamma_L^\mu \nu_{\mu m}$ or $\bar{e}_m\gamma_L^\mu \mu_m$.

2.4 The unitarization conditions

They are five: three arise from the absence of non-diagonal neutral currents for mass eigenstates, and two from the universality of diagonal currents. Accordingly, one degree of freedom is expected to be unconstrained.

2.4.1 Absence of non-diagonal neutral currents of mass eigenstates

The three conditions read:

* for the absence of $\{13\}$ and $\{31\}$ currents:

$$[13] = 0 = [31] \Leftrightarrow c_{12} [c_{13}s_{13} - \tilde{c}_{13}\tilde{s}_{13}(\tilde{c}_{23}^2 + s_{23}^2)] - \tilde{c}_{13}\tilde{s}_{12}(c_{23}s_{23} - \tilde{c}_{23}\tilde{s}_{23}) = 0; \quad (9)$$

* for the absence of $\{23\}$ and $\{32\}$ currents:

$$[23] = 0 = [32] \Leftrightarrow s_{12} [c_{13}s_{13} - \tilde{c}_{13}\tilde{s}_{13}(\tilde{c}_{23}^2 + s_{23}^2)] + \tilde{c}_{13}\tilde{c}_{12}(c_{23}s_{23} - \tilde{c}_{23}\tilde{s}_{23}) = 0; \quad (10)$$

* for the absence of $\{12\}$ and $\{21\}$ currents:

$$[12] = 0 = [21] \Leftrightarrow s_{12}c_{12}c_{13}^2 - \tilde{s}_{12}\tilde{c}_{12}(c_{23}^2 + \tilde{s}_{23}^2) + s_{12}c_{12}\tilde{s}_{13}^2(s_{23}^2 + \tilde{c}_{23}^2) + \tilde{s}_{13}(s_{12}\tilde{s}_{12} - c_{12}\tilde{c}_{12})(c_{23}s_{23} - \tilde{c}_{23}\tilde{s}_{23}) = 0. \quad (11)$$

2.4.2 Universality of diagonal neutral currents of mass eigenstates

The two independent conditions read:

* equality of $\{11\}$ and $\{22\}$ currents:

$$[11] - [22] = 0 \Leftrightarrow (c_{12}^2 - s_{12}^2) [c_{13}^2 + \tilde{s}_{13}^2(s_{23}^2 + \tilde{c}_{23}^2)] - (\tilde{c}_{12}^2 - \tilde{s}_{12}^2)(c_{23}^2 + \tilde{s}_{23}^2) + 2\tilde{s}_{13}(c_{23}s_{23} - \tilde{c}_{23}\tilde{s}_{23})(c_{12}\tilde{s}_{12} + s_{12}\tilde{c}_{12}) = 0; \quad (12)$$

* equality of $\{22\}$ and $\{33\}$ currents:

$$[22] - [33] = 0 \Leftrightarrow s_{12}^2 + \tilde{c}_{12}^2(c_{23}^2 + \tilde{s}_{23}^2) - (s_{23}^2 + \tilde{c}_{23}^2) + (1 + s_{12}^2) [\tilde{s}_{13}^2(s_{23}^2 + \tilde{c}_{23}^2) - s_{13}^2] + 2s_{12}\tilde{s}_{13}\tilde{c}_{12}(\tilde{c}_{23}\tilde{s}_{23} - c_{23}s_{23}) = 0. \quad (13)$$

The equality of $\{11\}$ and $\{33\}$ currents is of course not an independent condition.

2.5 Solutions for $\theta_{13} = 0 = \tilde{\theta}_{13}$

In a first step, to ease solving the system of trigonometric equations, we shall study the configuration in which one of the two angles parametrizing the 1-3 mixing vanishes⁵, which is very close to what is observed experimentally in the quark sector, and likely in the neutrino sector. It turns out, as demonstrated in Appendix A, that the second mixing angle vanishes simultaneously. We accordingly work in the approximation (the sensitivity of the solutions to a small variation of $\theta_{13}, \tilde{\theta}_{13}$ will be studied afterwards)

$$\theta_{13} = 0 = \tilde{\theta}_{13}. \quad (14)$$

Eqs. (9), (10), (11), (12) and (13), reduce in this limit to

⁵By doing so, we exploit the possibility to fix one degree of freedom left *a priori* unconstrained by the five equations; see subsection 2.2.

$$-\tilde{s}_{12}(c_{23}s_{23} - \tilde{c}_{23}\tilde{s}_{23}) = 0, \quad (15a)$$

$$\tilde{c}_{12}(c_{23}s_{23} - \tilde{c}_{23}\tilde{s}_{23}) = 0, \quad (15b)$$

$$s_{12}c_{12} - \tilde{s}_{12}\tilde{c}_{12}(c_{23}^2 + \tilde{s}_{23}^2) = 0, \quad (15c)$$

$$(c_{12}^2 - s_{12}^2) - (\tilde{c}_{12}^2 - \tilde{s}_{12}^2)(c_{23}^2 + \tilde{s}_{23}^2) = 0, \quad (15d)$$

$$s_{12}^2 + \tilde{c}_{12}^2(c_{23}^2 + \tilde{s}_{23}^2) - (s_{23}^2 + \tilde{c}_{23}^2) = 0. \quad (15e)$$

It is shown in Appendix B that the only solutions are θ_{12} and θ_{23} Cabibbo-like ($\tilde{\theta}_{12,23} = \theta_{12,23} + k\pi$) or maximal ($\theta_{12,23} = \pi/4 + n\pi/2$, $\tilde{\theta}_{12,23} = \pi/4 + m\pi/2$).

Accordingly, the two following sections will respectively start from:

* θ_{12} and θ_{23} Cabibbo-like (and, in a first step, vanishing θ_{13}), which finally leads to a mixing pattern similar to what is observed for quarks;

* θ_{23} maximal and θ_{12} Cabibbo like (and, in a first step, vanishing θ_{13}), which finally leads to a mixing pattern similar to the one observed for neutrinos.

3 The quark sector; constraining the three CKM angles

Mass splittings entail that the previous general conditions, which, when exactly satisfied, correspond *de facto* to unitary mixing matrices, cannot be exactly fulfilled. We investigate the vicinity of their solutions, and show that the same violation pattern that led to an accurate determination of the Cabibbo angle in the case of two generations, is also satisfied by the CKM angles in the case of three generations.

3.1 The simplified case $\theta_{13} = 0 = \tilde{\theta}_{13}$

In the neighborhood of the solution with both θ_{12} and θ_{23} Cabibbo-like, we write

$$\begin{aligned} \tilde{\theta}_{12} &= \theta_{12} + \epsilon, \\ \tilde{\theta}_{23} &= \theta_{23} + \eta. \end{aligned} \quad (16)$$

The pattern ($\theta_{13} = 0 = \tilde{\theta}_{13}$) can be reasonably considered to be close to the experimental situation, at least close enough for trusting not only the relations involving the first and second generation, but also the third one.

Like in [3], we impose that the absence of $\{12\}$, $\{21\}$ neutral currents is violated with the same strength as the universality of $\{11\}$ and $\{22\}$ currents. It reads

$$|2\eta s_{12}c_{12}s_{23}c_{23} + \epsilon(c_{12}^2 - s_{12}^2)| = |-2\eta s_{23}c_{23}(c_{12}^2 - s_{12}^2) + 4\epsilon s_{12}c_{12}|. \quad (17)$$

We choose the “+” sign for both sides, such that, for two generations only, the Cabibbo angle satisfies $\tan(2\theta_{12}) = +1/2$. (17) yields the ratio η/ϵ , that we then plug into the condition equivalent to (17) for the (2, 3) channel.

$$|\eta c_{12}(c_{23}^2 - s_{23}^2)| = |2\eta s_{23}c_{23}(1 + c_{12}^2) - 2\epsilon s_{12}c_{12}|. \quad (18)$$

(17) and (18) yield

$$\tan(2\theta_{23}) = \frac{c_{12}}{1 + c_{12}^2 - 2s_{12}c_{12} \frac{(s_{12}c_{12} + c_{12}^2 - s_{12}^2)}{4s_{12}c_{12} - (c_{12}^2 - s_{12}^2)}} \approx \frac{c_{12}}{2 - \frac{5}{4} \frac{s_{12}c_{12}}{\tan(2\theta_{12}) - \frac{1}{2}}}. \quad (19)$$

In the r.h.s. of (19), we have assumed that θ_{12} is close to its Cabibbo value $\tan(2\theta_{12}) \approx 1/2$. θ_{23} is seen to vanish with $[\tan(2\theta_{23}) - 1/2]$. The predicted value for θ_{23} is plotted in Fig. 1 as a function of θ_{12} , together with the experimental intervals for θ_{23} and θ_{12} . There are two [7] for θ_{12} ; the first comes from the measures of V_{ud} (in black on Fig. 1)

$$V_{ud} \in [0.9735, 0.9740] \Rightarrow \theta_{12} \in [0.2285, 0.2307], \quad (20)$$

and the second from the measures of V_{us} (in purple on Fig. 1)

$$V_{us} \in [0.2236, 0.2278] \Rightarrow \theta_{12} \in [0.2255, 0.2298]. \quad (21)$$

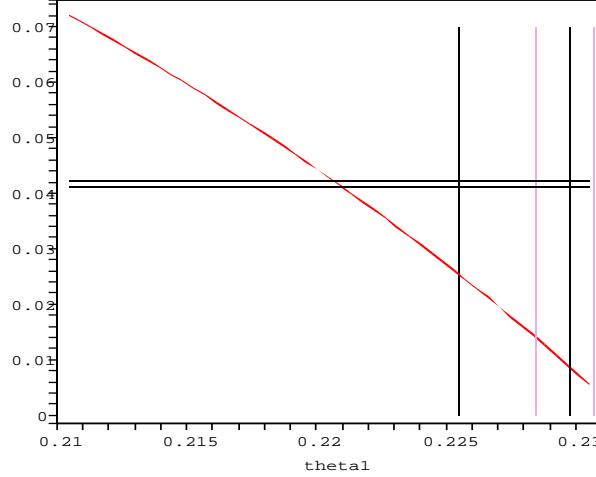


Fig. 1: θ_{23} for quarks as a function of θ_{12} ; simplified case $\theta_{13} = 0 = \tilde{\theta}_{13}$

The measured value for θ_{23} is seen on Fig. 1 to correspond to $\theta_{12} \approx 0.221$, that is $\cos(\theta_{12}) \approx 0.9757$. Our prediction for $\cos(\theta_{12})$ is accordingly $1.7 \cdot 10^{-3}$ away from the upper limit of the present upper bound for $V_{ud} \equiv c_{12}c_{13}$ [8] [7]; it corresponds to twice the experimental uncertainty. It also corresponds to $\sin(\theta_{12}) = 0.2192$, while $V_{us} \equiv s_{12}c_{13}$ is measured to be $0.2247(19)$ [9] [7]; there, the discrepancy is $2/100$, only slightly above the $1.8/100$ relative width of the experimental interval.

The approximation which sets $\theta_{13} = 0 = \tilde{\theta}_{13}$ is accordingly reasonable, though it yields results slightly away from experimental bounds. We show in the next subsection that relaxing this approximation gives results in excellent agreement with present experiments.

3.2 Going to $(\theta_{13} \neq 0, \tilde{\theta}_{13} \neq 0)$

Considering all angles to be Cabibbo-like with, in addition to (16)

$$\tilde{\theta}_{13} = \theta_{13} + \rho, \quad (22)$$

the l.h.s.'s of eqs. (9),(10),(11), (12), (13) and the sum (12 + 13) depart respectively from zero by

$$\eta c_{13} [s_{12}(c_{23}^2 - s_{23}^2) + 2s_{13}c_{12}c_{23}s_{23}] - \rho c_{12}(c_{13}^2 - s_{13}^2); \quad (23a)$$

$$\eta c_{13} [-c_{12}(c_{23}^2 - s_{23}^2) + 2s_{13}s_{12}c_{23}s_{23}] - \rho s_{12}(c_{13}^2 - s_{13}^2); \quad (23b)$$

$$-\epsilon(c_{12}^2 - s_{12}^2) + \eta [s_{13}(c_{23}^2 - s_{23}^2)(c_{12}^2 - s_{12}^2) - 2c_{23}s_{23}c_{12}s_{12}(1 + s_{13}^2)] + 2\rho c_{13}s_{13}c_{12}s_{12}; \quad (23c)$$

$$4\epsilon c_{12}s_{12} + \eta [-4s_{13}s_{12}c_{12}(c_{23}^2 - s_{23}^2) - 2c_{23}s_{23}(c_{12}^2 - s_{12}^2)(1 + s_{13}^2)] + 2\rho c_{13}s_{13}(c_{12}^2 - s_{12}^2); \quad (23d)$$

$$-2\epsilon s_{12}c_{12} + \eta [2s_{13}c_{12}s_{12}(c_{23}^2 - s_{23}^2) + 2c_{23}s_{23}((c_{12}^2 - s_{12}^2) + c_{13}^2(1 + s_{12}^2))] + 2\rho c_{13}s_{13}(1 + s_{12}^2); \quad (23e)$$

$$2\epsilon s_{12}c_{12} + \eta [-2s_{13}c_{12}s_{12}(c_{23}^2 - s_{23}^2) + 2c_{23}s_{23}(c_{13}^2(1 + c_{12}^2) - (c_{12}^2 - s_{12}^2))] + 2\rho c_{13}s_{13}(1 + c_{12}^2). \quad (23f)$$

We have added (23f), which is not an independent relation, but the sum of (23d) and (23e); it expresses the violation in the universality of diagonal $\{11\}$ and $\{33\}$ currents.

3.2.1 A guiding calculation

Before doing the calculation in full generality, and to make a clearer difference with the neutrino case, we first do it in the limit where one neglects terms which are quadratic in the small quantities θ_{13} and ρ . By providing simple intermediate formulæ, it enables in particular to suitably choose the signs which occur in equating the moduli of two quantities. Eqs.(23) become

$$\eta [s_{12}(c_{23}^2 - s_{23}^2) + 2s_{13}c_{12}c_{23}s_{23}] - \rho c_{12}; \quad (24a)$$

$$\eta [-c_{12}(c_{23}^2 - s_{23}^2) + 2s_{13}s_{12}c_{23}s_{23}] - \rho s_{12}; \quad (24b)$$

$$-\epsilon(c_{12}^2 - s_{12}^2) + \eta [s_{13}(c_{23}^2 - s_{23}^2)(c_{12}^2 - s_{12}^2) - 2c_{23}s_{23}c_{12}s_{12}]; \quad (24c)$$

$$4\epsilon c_{12}s_{12} - 2\eta [2s_{13}s_{12}c_{12}(c_{23}^2 - s_{23}^2) + c_{23}s_{23}(c_{12}^2 - s_{12}^2)]; \quad (24d)$$

$$-2\epsilon s_{12}c_{12} + 2\eta [s_{13}c_{12}s_{12}(c_{23}^2 - s_{23}^2) + c_{23}s_{23}(1 + c_{12}^2)]; \quad (24e)$$

$$2\epsilon s_{12}c_{12} + 2\eta [-s_{13}c_{12}s_{12}(c_{23}^2 - s_{23}^2) + c_{23}s_{23}(1 + s_{12}^2)]. \quad (24f)$$

The principle of the method is the same as before. From (24c) = -(24d)⁶, which expresses that the absence of non-diagonal $\{12\}$ current is violated with the same strength as the universality of $\{11\}$ and

⁶The (-) signs ensures that $\tan(2\theta_{12}) \approx (+)1/2$.

$\{22\}$ currents, one gets ϵ/η as a function of $\theta_{12}, \theta_{23}, \theta_{13}$ ⁷. This expression is plugged in the relation (24b) = (-)(24e)⁸, which expresses the same condition for the (2, 3) channel; from this, one extracts ρ/η as a function of $\theta_{12}, \theta_{23}, \theta_{13}$ ⁹. The expressions that have been obtained for ϵ/η and ρ/η are then inserted into the third relation, $|(24a)| = |(24f)|$, which now corresponds to the (1, 3) channel. This last step yields a relation $F_0(\theta_{12}, \theta_{23}, \theta_{13}) = 1$ between the three angles $\theta_{12}, \theta_{23}, \theta_{13}$.

It turns out that $\frac{\partial F_0(\theta_{12}, \theta_{23}, \theta_{13})}{\partial \theta_{13}} = 0$, such that, in this case, a condition between θ_{12} and θ_{23} alone eventually fulfills the three relations under concern

$$1 = \left| \frac{\text{viol}([11] = [22])}{\text{viol}([12] = 0 = [21])} \right| = \left| \frac{\text{viol}([22] = [33])}{\text{viol}([23] = 0 = [32])} \right| = \left| \frac{\text{viol}([11] = [33])}{\text{viol}([13] = 0 = [31])} \right| \Leftrightarrow \tilde{F}_0(\theta_{12}, \theta_{23}) = 1. \quad (27)$$

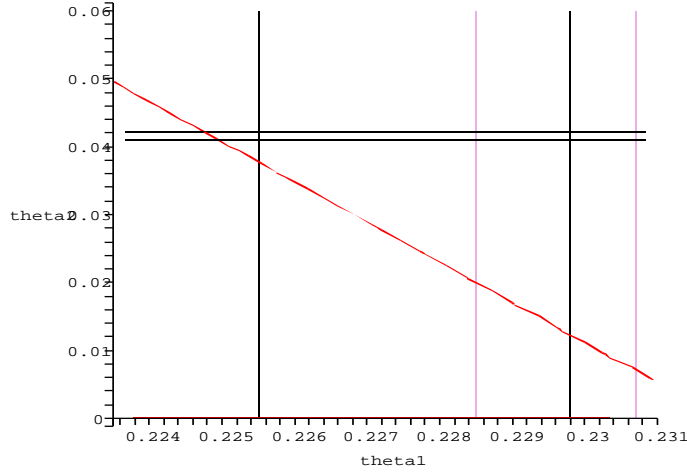


Fig. 2: θ_{23} for quarks as a function of θ_{12} ; neglecting terms quadratic in θ_{13}

θ_{23} is plotted on Fig. 2 as a function of θ_{12} , together with the experimental intervals for θ_{23} and θ_{12} (the intervals for θ_{12} come respectively from V_{ud} (eq. (20)) and V_{us} (eq. (21))).

The precision obtained is much better than in Fig. 1 since, in particular, for θ_{23} within its experimental range, the discrepancy between the predicted θ_{12} and its lower experimental limit coming from V_{us} is smaller than the two experimental intervals, and even smaller than their intersection.

3.2.2 The general solution

The principle for solving the general equations (23) is the same as above. One first uses the relation (23c) = (-) (23d) to determine ρ/ϵ in terms of η/ϵ . The result is plugged in the relation (23b) = (-) (23e), which fixes η/ϵ , and thus ρ/ϵ as functions of $(\theta_{12}, \theta_{23}, \theta_{13})$. These expressions for η/ϵ and ρ/ϵ are finally plugged in the relation $|(23a)| = |(23f)|$, which provides a condition $F(\theta_{12}, \theta_{23}, \theta_{13}) = 1$. When it is

7

$$\frac{\epsilon}{\eta} = s_{13}(c_{23}^2 - s_{23}^2) + 2s_{23}c_{23} \frac{s_{12}c_{12} + c_{12}^2 - s_{12}^2}{4c_{12}s_{12} - (c_{12}^2 - s_{12}^2)}; \quad (25)$$

ϵ/η has a pole at $\tan(2\theta_{12}) = 1/2$, the predicted value of the Cabibbo angle for two generations.

⁸There, again, the (-) sign has to be chosen so as to recover approximately (19).

9

$$\frac{\rho}{\eta} = 2c_{23}s_{23} \left[s_{13} - c_{12} \left(2 \frac{(c_{12}s_{12} + c_{12}^2 - s_{12}^2)}{4s_{12}c_{12} - (c_{12}^2 - s_{12}^2)} - \frac{1 + c_{12}^2}{c_{12}s_{12}} + \frac{1}{s_{12}} \frac{c_{23}^2 - s_{23}^2}{2s_{23}c_{23}} \right) \right]. \quad (26)$$

ρ/η has a pole at $\tan(2\theta_{12}) = 1/2$ and, for $\theta_{13} = 0$, it vanishes, as expected, when θ_{12} and θ_{23} satisfy the relation (19), which has been deduced for $\tilde{\theta}_{13}(\equiv \theta_{13} + \rho) = 0 = \theta_{13}$.

fulfilled, the universality of each pair of diagonal neutral currents of mass eigenstates and the absence of the corresponding non-diagonal currents are violated with the same strength, in the three channels (1, 2), (2, 3) and (1, 3).

The results are displayed in Fig. 3; θ_{23} is plotted as a function of θ_{12} for $\theta_{13} = 0.004$ and 0.01 . The present experimental interval is [7]

$$V_{ub} = \sin(\theta_{13}) \approx \theta_{13} \in [4 \cdot 10^{-3}, 4.6 \cdot 10^{-3}]. \quad (28)$$

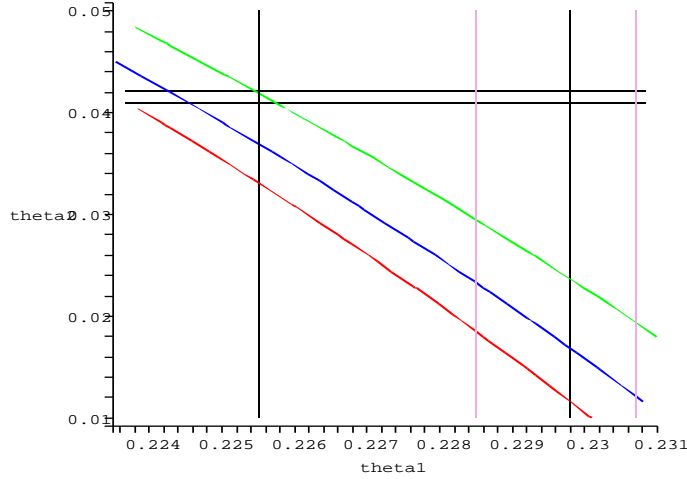


Fig. 3: θ_{23} for quarks as a function of θ_{12} , general case. $\theta_{13} = 0$ (red), 0.004 (blue) and 0.01 (green)

We conclude that:

- * The discrepancy between our predictions and experiments is smaller than the experimental uncertainty;
- * a slightly larger value of θ_{13} and/or slightly smaller values of θ_{23} and/or θ_{12} still increase the agreement between our predictions and experimental measurements;
- * the determination of θ_{12} from V_{us} seems preferred to that from V_{ud} .

Another confirmation of the relevance of our criterion is given in the next section concerning neutrino mixing angles.

4 A neutrino-like pattern; quark-lepton complementarity

In the “quark case”, we dealt with three “Cabibbo-like” angles. The configuration that we investigate here is the one in which θ_{23} is, as observed experimentally [7], (close to) maximal, and θ_{12} and θ_{13} are Cabibbo-like (see subsection 2.5).

4.1 The case $\theta_{13} = 0 = \tilde{\theta}_{13}$

We explore the vicinity of this solution, slightly departing from the corresponding unitary mixing matrix, by considering that $\tilde{\theta}_{12}$ now slightly differs from θ_{12} , and $\tilde{\theta}_{23}$ from its maximal value

$$\begin{aligned} \tilde{\theta}_{12} &= \theta_{12} + \epsilon, \\ \theta_{23} = \pi/4 &, \quad \tilde{\theta}_{23} = \theta_{23} + \eta. \end{aligned} \quad (29)$$

The l.h.s.’s of eqs. (9) (10) (11) (12) and (13) no longer vanish, and become respectively

$$-\frac{1}{2}\eta^2(s_{12} + \epsilon c_{12}), \quad (30a)$$

$$\frac{1}{2}\eta^2(c_{12} - \epsilon s_{12}), \quad (30b)$$

$$* - \eta s_{12} c_{12} + \epsilon(s_{12}^2 - c_{12}^2)(1 + \eta), \quad (30c)$$

$$* - \eta(c_{12}^2 - s_{12}^2) + 4\epsilon s_{12} c_{12}(1 + \eta), \quad (30d)$$

$$\eta(1 + c_{12}^2) - 2\epsilon s_{12} c_{12}(1 + \eta), \quad (30e)$$

showing by which amount the five conditions under scrutiny are now violated. Some care has to be taken concerning the accurateness of equations (30). Indeed, we imposed a value of θ_{13} which is probably not the physical one (even if close to). It is then reasonable to consider that channel $(1, 2)$ is the less sensitive to this approximation and that, accordingly, of the five equations above, (30c) and (30d), marked with an “*”, are the most accurate ¹⁰.

The question: is there a special value of $\theta_{12} = \tilde{\theta}_{12}$ Cabibbo-like for which small deviations (ϵ, η) from unitarity entail equal strength violations of

* the absence of $\{12\}, \{21\}$ non-diagonal neutral currents;

* the universality of $\{11\}$ and $\{22\}$ neutral currents ?

gets then a simple answer

$$s_{12} c_{12} = c_{12}^2 - s_{12}^2 \Rightarrow \tan(2\theta_{12}) = 2. \quad (31)$$

We did not take into account the terms proportional to ϵ because we assumed that the mass splittings between the first and second generations (from which the lack of unitarity originates) are much smaller than the ones between the second and the third generation ¹¹.

In the case of two generations, only ϵ appears, and one immediately recovers from (30c) and (30d) the condition fixing $\tan(2\theta_c) = 1/2$ for the Cabibbo angle.

Accordingly, the same type of requirement that led to a value of the Cabibbo angle for two generations very close to the observed value leads, for three generations, to a value of the first mixing angle satisfying the quark-lepton complementarity relation (3) [6].

The values of θ_{12} and θ_{23} determined through this procedure are very close to the observed neutrino mixing angles [7].

Though we only considered the two equations that are *a priori* the least sensitive to our choice of a vanishing third mixing angle (which is not yet confirmed experimentally), it is instructive to investigate the sensitivity of our solution to a small non-vanishing value of θ_{13} . This is done in Appendix C in which, for this purpose, we made the simplification $\tilde{\theta}_{13} \approx \theta_{13}$. It turns out that the terms proportional to s_{13} in the two equations $[12] = 0 = [21]$ and $|[11]| = |[22]|$ are also proportional to $(c_{23}^2 - s_{23}^2)$, such that our solution with θ_{23} maximal is very stable with respect to a variation of θ_{13} around zero. This may of course not be the case for the other three equations, which are expected to be more sensitive to the value of θ_{13} .

¹⁰The limitation of this approximation also appears in the fact that (30b), of second order in η , is not compatible with (30e), which is of first order.

¹¹Since the three angles play *a priori* symmetric roles, the simultaneous vanishing of θ and $\tilde{\theta}$, which we demonstrated for θ_{13} and $\tilde{\theta}_{13}$ (see Appendix A), should also occur for the other angles. Two competing effects accordingly contribute to the magnitude of the parameters $\epsilon, \eta \dots$: on one hand, they should be proportional to (some power of) the corresponding θ , and, on the other hand, one reasonably expects them to increase with the mass splitting between the fermions mixed by this θ . So, in the quark sector, that the violation of unitarity should be maximal for θ_{13} is not guaranteed since the corresponding mixing angle is also very small (as expected from hierarchical mixing matrices [10]). A detailed investigation of this phenomenon is postponed to a further work. In the neutrino sector, however, since θ_{23} is maximal (large), the assumption that the mass splitting between the second and third generation is larger than between the first and second is enough to guarantee $\epsilon \ll \eta$.

4.2 Prediction for θ_{13}

We now consider, like we did for quarks, the general case $\theta_{13} \neq 0 \neq \tilde{\theta}_{13}(\rho \neq 0)$, $\tilde{\theta}_{12} \neq \theta_{12}(\epsilon \neq 0)$, $\tilde{\theta}_{23} \neq \theta_{23}(\eta \neq 0)$, while assigning to θ_{12} and θ_{23} their values predicted in subsection 4.1.

We investigate the eight different relations between θ_{12} , θ_{23} and θ_{13} which originate from the $2 \times 2 \times 2$ possible sign combinations in the conditions (27) (the r.h.s. is now replaced by a condition $F(\theta_{12}, \theta_{23}, \theta_{13}) = 1$ involving the three mixing angles), where each modulus can be alternatively replaced by “+” or “-”.

Among the solutions found for θ_{13} , only two (up to a sign) satisfy the very loose experimental bound

$$\sin^2(\theta_{13}) \leq 0.1. \quad (32)$$

They correspond respectively to the sign combinations $(+/-/-)$, $(+/+/+)$, $(-/+ /+)$ and $(-/-/-)$

$$\begin{aligned} \theta_{13} &= \pm 0.2717 \quad , \quad \sin^2(\theta_{13}) = 0.072, \\ \theta_{13} &= \pm 5.7 \cdot 10^{-3} \quad , \quad \sin^2(\theta_{13}) = 3.3 \cdot 10^{-5}. \end{aligned} \quad (33)$$

The most recent experimental bounds can be found in [11]. They read

$$\sin^2(\theta_{13}) \leq 0.05, \quad (34)$$

which only leaves the smallest solution in (33) ¹².

Future experiments will confirm, or infirm, for neutrinos, the properties that we have shown to be satisfied with an impressive accuracy by quark mixing angles.

5 Comments and open issues

5.1 How close are mixing matrices to unitarity? Mixing angles and mass spectrum

An important characteristic of the conditions that fix the mixing angles is that they do not depend on the strength of the violation of the two properties under scrutiny, namely, the absence of non-diagonal neutral currents and the universality of the diagonal ones in the space of mass eigenstates. Since only their ratio is concerned, each violation can be infinitesimally small.

This is, on one side, fortunate since we have not yet been able to calculate the magnitude of the violation of the unitarity of the mixing matrices from, for example, mass ratios. The issue, for fundamental particles, turns indeed to be much more difficult conceptually than it was for composite particles like neutral kaons [1].

But, on the other side, this blurs the relation between the mixing pattern and the fermionic mass spectrum ¹³. This was already blatant with the emergence of maximal mixing as a special set of solutions of the unitarization equations in [2], which did not depend of any special type of mass hierarchy. The question now arises of finding, if any, properties of the mass spectrum, which are, through the products $K^\dagger K$ of mixing matrices, compatible with an $SU(2)$ symmetric pattern of weak neutral currents.

¹²Our predictions substantially differs from the ones in [12], which mainly focuses on special textures for the product of the quark and neutrino mixing matrices [13].

¹³A rigorous investigation of this connection was done in [10]. It however rests on the assumption (incorrect in QFT) that a system of coupled fermions can be described by a unique constant mass matrix, which is diagonalized by a bi-unitary transformation. Then the so-defined “fermion masses” are not the eigenvalues of the mass matrix, which makes all the more tricky the connection with the poles of the full propagator in QFT.

5.2 Which mixing angles are measured

The results that have been exposed are valid for fermions of both electric charges. They concern the mixing angles which parametrize

- * for quarks, the mixing matrix K_u of u -type quarks as well as K_d of d -type quarks;
- * for leptons, the mixing matrix K_ν of neutrinos as well as that of charged leptons K_ℓ ,

and we have shown that our approach allows to obtain on purely theoretical grounds the values of the mixing angles which are experimentally determined.

However, a problem arises : the measured values of the mixing angles are commonly attached, not to a single mixing matrix, e.g. K_u or K_d , but to the product $K = K_u^\dagger K_d$ which occurs in charged currents when both quark types are mass eigenstates. Thus, in the standard approach, they are *a priori* related to an entanglement of the mixing angles of quarks (or leptons) of different charges. This problem gets easily solved by the following argumentation. Considering, for example, semi-leptonic decays of pseudoscalar mesons in the approach where one of the constituent quarks is “spectator”, we show that only one of the two mixing matrices is involved and measured. Indeed, while the two-fold nature (flavor and mass) of the neutral kaons has always been acknowledged, this step has never been taken yet for other mesons. This is what we do now, in a rather naive, but efficient way, that consists of distinguishing a $[\bar{q}_{i,f} q_{j,f}]$ “flavor” meson from the mass eigenstate $[\bar{q}_{i,m} q_{j,m}]$ ($q_{i,j}$ being the constituent quarks). Consider for example, the decay $K^0 \rightarrow \pi^- e^+ \nu_e$. The K^0 that decay semi-leptonically being produced by strong interactions cannot be but a flavor meson $[\bar{s}_f d_f]$, while its decay product π^- , which is identified by its mass and charge, is a mass eigenstate $[\bar{u}_m d_m]$. At the quark level, the weak transition occurs accordingly between a flavour eigenstate (\bar{s}_f) to a mass eigenstate (\bar{u}_m), which only involves *one* mixing matrix, K_u , and not the product $K_u^\dagger K_d$. As for the spectator quark, the transition from its flavor state d_f to its mass state d_m involves the cosine of the corresponding mixing angle, which is always close to 1. It thus appears that the mixing angles that are measured in such processes are the ones of K_u or K_d (up to a cosine factor very close to 1), which fits with our symmetric prediction.

The same problem is expected in the leptonic sector. Its solution depends on the nature of the neutrino eigenstates that are produced and detected. Let us consider for example the case of solar neutrinos. If the flux predicted in solar models concerns flavour neutrinos, and if the detection process also counts flavour neutrinos, the sole mixing matrix which controls their evolution and oscillations is K_ν , because it is the only matrix involved in the projection of flavour states onto mass states. This is the most likely situation. Indeed, the production mechanism inside the sun occurs through nuclear beta decay, in which the protons and neutrons, being bound by strong forces, are presumably, together with their constituent quarks, flavour eigenstates. The detection (for example the transition from chlorine to argon) also occurs through nuclear (inverse) beta decay, which accordingly also counts the number of ν_{ef} reaching the detector. The situation would be different if the comparison was made between the fluxes of the eigenstates ν_e, ν_μ, ν_τ defined in subsection 2.1 (see also appendix D); since their projections on the mass eigenstates involve the product $K_\ell^\dagger K_\nu$, their oscillations are now controlled by an entanglement of the mixing angles of neutrinos and charged leptons.

5.3 A multiscale problem

Recovery of the present results by perturbative techniques (Feynman diagrams) stays an open issue. All the subtlety of the problem lies in the inadequacy of using a single constant mass matrix; because non-degenerate coupled systems are multiscale systems, as many mass matrix should be introduced as there are poles in the (matricial) propagator [15]¹⁴.

¹⁴In QFT, as opposed to a Quantum Mechanical treatment (in which a single constant mass matrix is introduced – this is the Wigner-Weisskopf approximation–), a constant mass matrix can only be introduced in a linear approximation to the inverse propagator in the vicinity of each of its poles [1]. When several coupled states are concerned, the (matricial) propagator having several poles, as many (constant) mass matrices should be introduced [15]; only one of the eigenstates of each of these mass matrices corresponds to a physical (mass) eigenstate.

The existence of different scales makes the use of an “on-shell” renormalized Lagrangian [16] hazardous, because each possible renormalization scale optimizes the calculation of parameters at this scale, while, for other scales, one has to rely on renormalization group equations.

Unfortunately, these equations have only been approximately solved with the simplifying assumption that the renormalized mass matrices are hermitian¹⁵ and that the renormalized mixing matrices are unitary [16]. Performing the same job dropping these hypotheses looks rather formidable and beyond the scope of the present work. It also unfortunately turns out that, as far as the Yukawa couplings are concerned, the expressions that have been obtained at two loops for their β functions (which start the evolution only up from the top quark mass) [17] have poles in $(m_i - m_j)$, which makes them inadequate for the study of subsystems with masses below the top quark mass.

5.4 Using a q^2 -dependent renormalized mass matrix

Departure from the inappropriate Wigner-Weisskopf approximation can also be done by working with an effective renormalized q^2 -dependent mass matrix $M(q^2)$. It however leads to similar conclusions as the present approach.

Its eigenvalues are now q^2 -dependent, and are determined by the equation $\det[M(q^2) - \lambda(q^2)] = 0$ ¹⁶. Let them be $\lambda_1(q^2) \dots \lambda_n(q^2)$. The physical masses satisfy the n self-consistent equations $q^2 = \lambda_{1\dots n}(q^2)$, such that $m_1^2 = \lambda_1(m_1^2) \dots m_n^2 = \lambda_n(m_n^2)$. At each m_i^2 , $M(m_i^2)$ has n eigenvectors, but only one corresponds to the physical mass eigenstate; the others are “spurious” states [1]. Even if the renormalized mass matrix is hermitian at any given q^2 , the physical mass eigenstates corresponding to different q^2 belong to as many different orthonormal sets of eigenstates and thus, in general, do not form an orthonormal set. The discussion proceeds like in the core of the paper.

Determining the exact form of the renormalized mass matrix could accordingly be a suitable way to recover our predictions via perturbative techniques (like was done in [1] for the quantitative prediction of the ratio ϵ_S/ϵ_L). As already mentioned, the difficulty is that hermiticity assumptions should be dropped, which open the possibility of departing from the unitarity of the mixing matrix. This is currently under investigation.

6 Conclusion and perspective

This work does not, obviously, belong to what is nowadays referred to as “Beyond the Standard Model”, since it does not incorporate any “new physics” such as supersymmetry, “grand unified theories (GUT)” or extra-dimensions. However it does not strictly lie within the SM either, even if it is very close to. Of course, it shares with the latter its general framework (mathematical background and physical content), and also borrows from it the two physical conditions of universality for diagonal neutral currents and absence of FCNC’s, which play a crucial role in the process. But, on the basis of the most general arguments of QFT, we make a decisive use of the essential non-unitarity of the mixing matrices, whereas only unitary matrices are present in the SM. This property may be considered, in the SM, as an “accidental” characteristic of objects which are intrinsically non-unitary.

The mixing angles experimentally observed get constrained in the vicinity of this “standard” situation, a slight departure from which being due to mass splittings. Hence our approach can be considered to explore the “Neighborhood of the Standard Model”, which is likely to exhibit low-energy manifestations of physics “Beyond the Standard Model”.

¹⁵One can go to hermitian mass matrices by rotating right-handed fermions *as far as they are not coupled*; however, at two loops, the charged weak currents also involve right-handed fermions, which cannot be anymore freely rotated.

¹⁶This is the simple case of a normal mass matrix, which can be diagonalized by a single (q^2 -dependent) unitary matrix. When it is non-normal, the standard procedure uses a bi-unitary diagonalization (see footnote 13).

While common approaches limit themselves to guessing symmetries for the mass matrices (see for example [18] and references therein), we showed that special patterns are instead likely to reveal themselves in the violation of some (wrongly) intuitive properties¹⁷. In each given (i, j) channel of mass eigenstates, the characteristic pattern that emerges is that two *a priori* different violations follow from a precise horizontal continuous symmetry, which is the most intuitive $SU(2)$ group attached to this pair of fermions. One simply falls back on an, up to now unraveled, manifestation of “old Current Algebra” [4]. It is remarkable that the same symmetry underlies both the quark and leptonic sectors, which was never suspected before; they only differ through the 0th order solution to the unitarization equations, the two-foldness of which was recently uncovered in [2]. We have in particular learned that symmetries relevant for flavour physics should not be looked for, or implemented, at the level of the mass matrices and Yukawa couplings, but at the level of the weak currents.

We have also argued that, unlike what is generally assumed, the mixing angles that are measured are (up to a cosine) the ones of a single mixing matrix, and not of the product $K_u^\dagger K_d$ or $K_\ell^\dagger K_\nu$. Our scheme then appears entirely coherent, and agrees with experimental data.

To conclude, the present work demonstrates that flavor physics satisfies very simple criteria which had been, up to now, unnoticed. Strong arguments have been presented in both the quark and leptonic sectors, which will be further tested when the third mixing angle of neutrinos is accurately determined.

Acknowledgments: *Discussions with A. Djouadi, J. Orloff and M.I. Vysotsky are gratefully acknowledged.*

¹⁷For a (constant unique) mass matrix, unitarity of the mixing matrix has commonly been linked with the unitarity of the theory. See also footnote 1.

Appendix

A $\tilde{\theta}_{13} = 0 \Rightarrow \theta_{13} = 0$

Using the notations of section 2, we start with the following system of equations:

$$\frac{[11] + [22]}{2} = [33] \Leftrightarrow s_{13}^2 + s_{23}^2 + \tilde{c}_{23}^2 = 1; \quad (35a)$$

$$[11] = [22] \Leftrightarrow c_{13}^2 \cos(2\theta_{12}) = (c_{23}^2 + \tilde{s}_{23}^2) \cos(2\tilde{\theta}_{12}); \quad (35b)$$

$$[12] = 0 = [21] \Leftrightarrow c_{13}^2 \sin(2\theta_{12}) = (c_{23}^2 + \tilde{s}_{23}^2) \sin(2\tilde{\theta}_{12}); \quad (35c)$$

$$[13] = 0 = [31] \Leftrightarrow \tilde{s}_{12} (\sin(2\theta_{23}) - \sin(2\tilde{\theta}_{23})) = c_{12} \sin(2\theta_{13}); \quad (35d)$$

$$[23] = 0 = [32] \Leftrightarrow \tilde{c}_{12} (\sin(2\tilde{\theta}_{23}) - \sin(2\theta_{23})) = s_{12} \sin(2\theta_{13}). \quad (35e)$$

From equation (35a), we have $c_{23}^2 + \tilde{s}_{23}^2 \neq 0$, which entails $c_{13}^2 \neq 0$ ¹⁸. Let us study the consequence on the two equations (35b) and (35c).

- the two sides of (35b) vanish for $\cos(2\theta_{12}) = 0 = \cos(2\tilde{\theta}_{12})$, i.e. $\theta_{12} = \frac{\pi}{4}[\frac{\pi}{2}] = \tilde{\theta}_{12}$. (35c) then gives $c_{13}^2 = c_{23}^2 + \tilde{s}_{23}^2$, which, associated with (35a), yields the following solution¹⁹: $\theta_{13} = 0[\pi]$ and $\tilde{\theta}_{23} = \pm\theta_{23}[\pi]$.
- the two sides of (35c) vanish for $\sin(2\theta_{12}) = 0 = \sin(2\tilde{\theta}_{12}) = 0$, i.e. $\theta_{12} = 0[\frac{\pi}{2}] = \tilde{\theta}_{12}$. (35b) gives then $c_{13}^2 = c_{23}^2 + \tilde{s}_{23}^2$, hence, like previously, $\theta_{13} = 0[\pi]$ and $\tilde{\theta}_{23} = \pm\theta_{23}[\pi]$.
- in the other cases we can calculate the ratio (35b) / (35c), which gives $\tan(2\theta_{12}) = \tan(2\tilde{\theta}_{12})$, hence $\theta_{12} = \tilde{\theta}_{12}[\pi]$ or $\theta_{12} = \frac{\pi}{2} + \tilde{\theta}_{12}[\pi]$:
 - * $\theta_{12} = \frac{\pi}{2} + \tilde{\theta}_{12}[\pi]$ implies for (35b)(35c) $c_{13}^2 = -c_{23}^2 - \tilde{s}_{23}^2$, which, together with (35a) ($c_{13}^2 = s_{23}^2 + \tilde{c}_{23}^2$), gives a contradiction : $2 = 0$:
 - * $\theta_{12} = \tilde{\theta}_{12}(\neq 0)[\pi]$ implies, like previously, $c_{13}^2 = c_{23}^2 + \tilde{s}_{23}^2$, which gives, when combined with (35a): $\theta_{13} = 0[\pi]$ and $\theta_{23} = \pm\theta_{23}[\pi]$.

Hence, it appears that whatever the case, the solution gives rise to $\theta_{13} = 0[\pi]$.

Let us now look at (35d) and (35e). Since $\theta_{13} = 0$, the two r.h.s.'s vanish, and we obtain the twin equations $\tilde{s}_{12}(\sin(2\theta_{23}) - \sin(2\tilde{\theta}_{23})) = 0$ and $\tilde{c}_{12}(\sin(2\theta_{23}) - \sin(2\tilde{\theta}_{23})) = 0$, which, together, imply $\sin(2\theta_{23}) = \sin(2\tilde{\theta}_{23})$. It follows that, either $\theta_{23} = \tilde{\theta}_{23}[\pi]$ or $\theta_{23} = \frac{\pi}{2} - \tilde{\theta}_{23}[\pi]$;

* $\theta_{23} = \tilde{\theta}_{23}[\pi]$ matches the result of the previous discussion in the “+” case, whereas, in the “-” case, the matching leads to $\theta_{23} = \tilde{\theta}_{23} = 0$, which is to be absorbed as a particular case in the “+” configuration;

* $\theta_{23} = \frac{\pi}{2} - \tilde{\theta}_{23}[\pi]$ matches the result of the previous discussion in the “+” configuration, in which case it leads to $\theta_{23} = \tilde{\theta}_{23} = \frac{\pi}{4}[\frac{\pi}{2}]$, i.e. maximal mixing between the fermions of the second and third generations.

¹⁸Indeed, let us suppose that c_{13} vanishes. Then $\cos(2\tilde{\theta}_{12})$ and $\sin(2\tilde{\theta}_{12})$ must vanish simultaneously, which is impossible.

¹⁹
$$\begin{cases} c_{13}^2 = c_{23}^2 + \tilde{s}_{23}^2 \\ s_{13}^2 + s_{23}^2 + \tilde{c}_{23}^2 = 1 \end{cases} \implies \begin{cases} s_{23}^2 + \tilde{c}_{23}^2 = 1 \\ s_{13}^2 = 0 \end{cases}$$

B $(\theta_{12}, \theta_{23})$ solutions of eqs. (9) (10) (11) (12) (13) for $\theta_{13} = 0 = \tilde{\theta}_{13}$

Excluding $\tilde{\theta}_{12} = 0$, (15a) and (15b) require $\sin(2\theta_{23}) = \sin(2\tilde{\theta}_{23}) \Rightarrow \tilde{\theta}_{23} = \theta_{23} + k\pi$ or $\tilde{\theta}_{23} = \pi/2 - \theta_{23} + k\pi$.

- for $\tilde{\theta}_{23} = \theta_{23} + k\pi$ Cabibbo-like,

(15c) requires $\sin(2\theta_{12}) = \sin(2\tilde{\theta}_{12}) \Rightarrow \tilde{\theta}_{12} = \theta_{12} + n\pi$ or $\tilde{\theta}_{12} = \pi/2 - \theta_{12} + n\pi$;

(15d) requires $\cos(2\theta_{12}) = \cos(2\tilde{\theta}_{12}) \Rightarrow \tilde{\theta}_{12} = \pm\theta_{12} + p\pi$;

(15e) requires $s_{12}^2 + \tilde{c}_{12}^2 - 1 = 0 \Rightarrow \tilde{\theta}_{12} = \pm\theta_{12} + r\pi$.

The solutions of these three equations are $\theta_{12} = \tilde{\theta}_{12} + k\pi$ Cabibbo-like or $\theta_{12} = \pi/4 + q\pi/2$ maximal.

- for $\tilde{\theta}_{23} = \pi/2 - \theta_{23} + k\pi$,

(15c) requires $s_{12}c_{12} = 2c_{23}^2\tilde{s}_{12}\tilde{c}_{12}$;

(15d) requires $c_{12}^2 - s_{12}^2 = 2c_{23}^2(\tilde{c}_{12}^2 - \tilde{s}_{12}^2)$;

(15e) requires $s_{12}^2 + 2c_{23}^2\tilde{c}_{12}^2 - 2s_{23}^2 = 0$.

The first two conditions yield $\tan(2\theta_{12}) = \tan(2\tilde{\theta}_{12}) = 2c_{23}^2 \Rightarrow \tilde{\theta}_{12} = \theta_{12} + k\pi/2 + n\pi$, which entails $2c_{23}^2 = 1 \Rightarrow \theta_{23} = \pm\pi/4 + p\pi/2$ maximal; $\tilde{\theta}_{23}$ is then maximal, too, and the third condition is automatically satisfied.

$\tilde{\theta}_{12} = \theta_{12} + n\pi$ is Cabibbo-like, while, for $\tilde{\theta}_{12} = \theta_{12} + (2k+1)\pi/2$, the second condition becomes $(c_{12}^2 - s_{12}^2) = 0$, which means that θ_{12} must be maximal.

C Sensitivity of the neutrino solution to a small variation of θ_{13}

If one allows for a small $\theta_{13} \approx \tilde{\theta}_{13}$, (11) and (12) become

$$\begin{aligned} & -2\eta s_{12}c_{12}s_{23}c_{23} + \epsilon(s_{12}^2 - c_{12}^2) + \eta s_{13}(c_{23}^2 - s_{23}^2)(c_{12}^2 - s_{12}^2), \\ & -2\eta s_{23}c_{23}(c_{12}^2 - s_{12}^2) + 4\epsilon s_{12}c_{12} - 2\eta s_{13}(c_{23}^2 - s_{23}^2)(2s_{12}c_{12} + \epsilon(c_{12}^2 - s_{12}^2)). \end{aligned} \quad (36)$$

For $\theta_{23}, \tilde{\theta}_{23}$ maximal, the dependence on θ_{13} drops out.

D Charged weak currents

Charged weak currents can be written in six different forms that are all strictly equivalent, but nonetheless refer to different physical pictures. As an example, for two generations of leptons :

$$\begin{aligned} & \overline{\begin{pmatrix} \nu_{ef} \\ \nu_{\mu f} \end{pmatrix}} W_\mu^+ \gamma_L^\mu \begin{pmatrix} e_f^- \\ \mu_f^- \end{pmatrix} = \overline{\begin{pmatrix} \nu_{em} \\ \nu_{\mu m} \end{pmatrix}} W_\mu^+ \gamma_L^\mu K_\nu^\dagger K_\ell \begin{pmatrix} e_m^- \\ \mu_m^- \end{pmatrix} \\ & = \overline{\begin{pmatrix} \nu_{ef} \\ \nu_{\mu f} \end{pmatrix}} W_\mu^+ \gamma_L^\mu \left[K_\ell \begin{pmatrix} e_m^- \\ \mu_m^- \end{pmatrix} \right] = \left[K_\nu \begin{pmatrix} \nu_{em} \\ \nu_{\mu m} \end{pmatrix} \right] W_\mu^+ \gamma_L^\mu \begin{pmatrix} e_f^- \\ \mu_f^- \end{pmatrix} \\ & = \overline{\begin{pmatrix} \nu_{em} \\ \nu_{\mu m} \end{pmatrix}} W_\mu^+ \gamma_L^\mu \left[K_\nu^\dagger \begin{pmatrix} e_f^- \\ \mu_f^- \end{pmatrix} \right] = \left[K_\ell^\dagger \begin{pmatrix} \nu_{ef} \\ \nu_{\mu f} \end{pmatrix} \right] W_\mu^+ \gamma_L^\mu \begin{pmatrix} e_m^- \\ \mu_m^- \end{pmatrix}. \end{aligned} \quad (37)$$

In the case where one of the $SU(2)$ partners, for example the charged lepton, is undoubtedly a mass eigenstate²⁰, the last expression of (37) shows that it is coupled to the so-called *electronic and muonic neutrinos*

$$\nu_e = (K_\ell^\dagger K_\nu)_{11} \nu_{em} + (K_\ell^\dagger K_\nu)_{12} \nu_{\mu m} = K_{\ell,11}^\dagger \nu_{ef} + K_{\ell,12}^\dagger \nu_{\mu f},$$

²⁰This is the case inside the sun [5] where, because of the limited available energy, only massive electrons can be produced, and also in the detection process of neutrinos on earth, which always proceeds via charged currents and the detection of produced physical (mass eigenstates) charged leptons.

$$\nu_\mu = (K_\ell^\dagger K_\nu)_{21} \nu_{em} + (K_\ell^\dagger K_\nu)_{22} \nu_{\mu m} = K_{\ell,21}^\dagger \nu_{ef} + K_{\ell,22}^\dagger \nu_{\mu f}. \quad (38)$$

The latter are neither flavour eigenstates, nor mass eigenstates, but a third kind of neutrinos, precisely defined as the ones which couple to electron and muon mass eigenstates in the weak charged currents

$$\overline{\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}} W_\mu^+ \gamma_L^\mu \begin{pmatrix} e_m^- \\ \mu_m^- \end{pmatrix}. \quad (39)$$

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